ACOUSTIC WAVES IN MULTI-LAYERED MEDIUM: II-ORDER PERTURBATION NUMBERS APPROACH

Jerzy SKRZYPCZYK [a], Agnieszka WINKLER-SKALNA[b]

[a] Faculty of Civil Engineering, The Silesian University of Technology, Akademicka 5, 44-100 Gliwice, Poland
E-mail address: jerzy.skrzypczyk@polsl.pl

[b] Faculty of Civil Engineering, The Silesian University of Technology, Akademicka 5, 44-100 Gliwice, Poland
E-mail address: agnieszka.winkler@polsl.pl

Received: 18.02.2008; Revised: 22.03.2008; Accepted: 19.05.2008

Abstract

A layered medium of finite height in which the n layers are assumed to have position dependent properties is considered. In the paper the boundary conditions are considered to be either rigid (Dirichlet-type) or of a reflecting type (mixed-type). The perturbation method based on II-order perturbation numbers is used to obtain the eigenvalues and the eigenfunctions of the height equation in case of both the rigid and reflecting boundaries. The corrections to the eigenvalues and eigenfunctions are numerically computed from the perturbation formulae in both cases of interest.

Streszczenie

Analizowany jest n-warstwowy ośrodek o skończonej wysokości (grubości), w którym parametry warstw zależą od położenia. Rozpatrywane są warunki brzegowe typu Dirichleta (brzeg sztywny) lub typu mieszanego (brzeg odbijający). Zastosowano metodę perturbacji, która wykorzystuje pojęcie liczb perturbacyjnych II rzędu. Otrzymano perturbacyjne wielkości wartości własnych i wektorów własnych równania opisującego położenie w obu rozpatrywanych przypadkach. Poprawki perturbacyjne dla wartości własnych i wektorów własnych zostały wyliczone numerycznie z równań perturbacyjnych dla rozpatrywanych przypadków.

Keywords: **Perturbation numbers; Multilayered medium; Wave propagation; Perturbed parameters.**

1. INTRODUCTION

The study of acoustic wave propagation in the nonhomogeneous medium attracted considerable attention in the past. It was motivated by the need to understand signal detection in acoustics and seismology.

In the paper we use special numbers, called perturbations numbers of the II-order. Recall that they are defined as ordered triples of real numbers $(x,y,z)\in R^3$ with specially defined algebra. [1-2]

It is however well known that the sound speed can vary due to variation in the temperature, humidity, salinity, location etc. This variation of the sound speed, however small, significantly affects the propagation of sound in the medium. [3-5]

The Helmholtz equation in terms of cylindrical coordinates is satisfied by the acoustic pressure $p^{(j)}$ in the *j*-th layer assuming radial symmetry. In our studies the bottom of the cylindrical domain is conveniently assumed to be rigid (Dirichlet conditions – I case) so that the height equation of the resulting problem has a simple behaviour. In practical situations, however, the boundary is not rigid but satisfies the absorbing or impedance boundary conditions (mixed-type conditions – II case). Physically, it means that part of the acoustic field is reflected while the other part of it is absorbed.

2. MATHEMATICAL MODEL

We consider here the N-layers model of the medium, similar to the discussed one in [3, 5]. Geometry of such model is shown in Fig. 1.

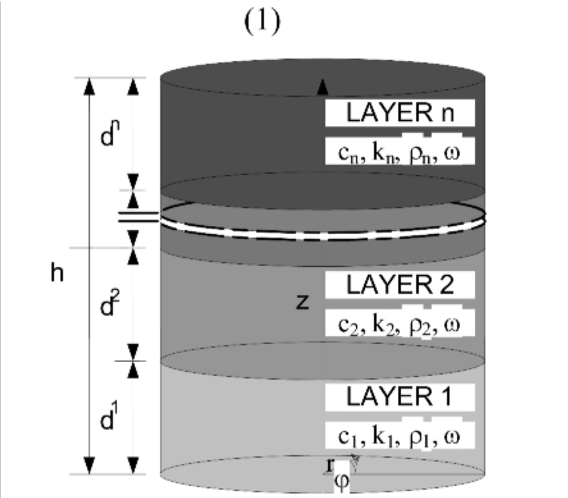

Figure 1.
Geometry of the problem and coordinate system

The analyzed medium of height h consists of N layers of uniform height d_i, $i=1,2,\ldots,N$. The lower layer has height d_1 so that the upper layer has height d_N. The boundary conditions are considered to be at $z=z_0=0$, $z=z_k:=\sum_{i=1}^{k} d_i$, $k=1,2,\ldots,N\text{-}1$ and $z=z_N=h$. The acoustic pressure $p^{(j)}$, density ρ_j, velocity c_j and the wave number k_j refer to these quantities in the j-th layer.

The Helmholtz equation is analyzed in terms of cylindrical coordinates (see Fig.1) and is satisfied by the acoustic pressure $p^{(j)}$ in the j-th layer:

$$\frac{\partial^2 p^{(i)}}{\partial r^2}+\frac{1}{r}\frac{\partial p^{(i)}}{\partial r}+\frac{\partial^2 p^{(i)}}{\partial z^2}+k_i^2 p^{(i)}=0 \quad k_i=\frac{\omega}{c_i},\ i=1,2,\ldots,N \tag{1}$$

By applying the separation variables method, i.e. assuming that $p^{(i)}(r,z)=R(r)\varphi^{(i)}(z)$, we get

$$\frac{d^2\varphi^{(i)}}{dz^2}+\left(k_i^2-\lambda\right)\varphi^{(i)}(z)=0, \qquad i=1,2,\ldots,N \tag{2}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\lambda R=0 \tag{3}$$

We assume radial symmetry of solutions of eqs. (1).

In our studies the bottom of the cylindrical domain is conveniently assumed to be free and the upper side of the cylinder is rigid (Dirichlet conditions – I case) or reflexive (mixed type conditions – II case) so that the height equation of the resulting problem can be obtained in the analytical way.

2.1. Rigid upper side of cylindrical domain

The bottom of the cylinder is assumed to be free, so we have

$$\varphi^{(1)}(0)=0. \tag{4}$$

Continuity of acoustic pressure at the interface $z=z_k:=\sum_{i=1}^{k} d_i$

$$\varphi^{(k)}(z_k)=\varphi^{(k+1)}(z_k), \qquad k=1,2,\ldots,N\text{-}1 \tag{5}$$

and continuity of the gradient of acoustic pressure gives

$$\frac{1}{\rho_k}\frac{d\varphi^{(k)}(z_k)}{dz}=\frac{1}{\rho_{k+1}}\frac{d\varphi^{(k+1)}(z_k)}{dz}, \qquad k=1,2,\ldots,N\text{-}1 \tag{6}$$

rigid bottom at the upper side $z=z_N=h$ gives

$$\frac{d\varphi^{(N)}(h)}{dz}=0 \tag{7}$$

For convenience, let $\gamma_i^2=k_i^2-\lambda$, $i=1,2,\ldots,N$, then eqs. (2) take the following form

$$\frac{d^2\varphi^{(i)}}{dz^2}+\gamma_i^2\varphi^{(i)}(z)=0, \qquad i=1,2,\ldots,N \tag{8}$$

The general solutions of eqs. (8) are given by

$$\varphi^{(i)}(z)=A_i\sin(\gamma_i z)+B_i\cos(\gamma_i z), \qquad z_i\le z\le z_{i+} \tag{9}$$

Using the boundary conditions (4)-(7), we obtain

$$B_1=0 \tag{10}$$

$$A_1\sin(\gamma_1 d_1)=A_2\sin(\gamma_2 d_1)+B_2\cos(\gamma_2 d_1) \tag{11}$$

$$\rho_{k+1}\gamma_k(A_k\cos(\gamma_k z_k)-B_k\sin(\gamma_k z_k))=\rho_k\gamma_{k+1}(A_{k+1}\cos(\gamma_{k+1}z_k)- -B_{k+1}\sin(\gamma_{k+1}z_k)),\ k=1,2,\ldots,N\text{-}1 \tag{12}$$

$$\gamma_N\left(A_N\cos(\gamma_N h)-B_N\sin(\gamma_N h)\right)=0 \tag{13}$$

Eqs. (10)-(13) form a homogeneous system of algebraic equations with $2N$ unknowns A_k, B_k, $k=1,2,\ldots,N$. It has a nontrivial solution if and only if the determinant of the coefficient matrix equals zero.

We get the whole family of eigenvalues λ_m, $m=1,2,3,\ldots$. Following that fact, the appropriate notation will be used further $\gamma_{im}^2 = k_i^2 - \lambda_m$, $i=1,2,\ldots,N$, $m=1,2,3,\ldots$. The corresponding eigenfunctions are given by

$$\varphi_m^{(1)}(z) = A_1 \sin(\gamma_{1m} z), \qquad 0 \le z \le z_1 = d_1 \tag{14}$$

$$\varphi_m^{(k)}(z) = A_k \sin(\gamma_{km} z) + B_k \cos(\gamma_{km} z), \quad z_{k-1} \le z \le z_k,\ k=2,3,..N \tag{15}$$

2.2. Reflecting upper side of cylindrical domain

We now consider another problem of the medium consisting of N homogeneous layers bounded by a pressure release surface at the bottom and a reflecting type upper side. The differential equations for the pressure are the same (2). The boundary conditions at $z=0$ and $z=z_k$, $k=1,2,\ldots,N-1$ are the same as in the previous case (see eqs. (4)-(6)). The boundary condition at the reflecting surface $z=h$ takes the form

$$\frac{d\varphi^{(N)}(h)}{dz} + \alpha \varphi^{(N)}(h) = 0 \tag{16}$$

Notice that if $\alpha=0$, then the reflecting condition becomes the rigid one, cf. eq. (7). Using the boundary conditions (4)-(6) and (16) one can obtain

$$B_1 = 0, \tag{17}$$

$$A_1 \sin(\gamma_1 d_1) = A_2 \sin(\gamma_2 d_1) + B_2 \cos(\gamma_2 d_1) \tag{18}$$

$$\rho_{k+1}\gamma_k(A_k \cos(\gamma_k z_k) - B_k \sin(\gamma_k z_k)) = \rho_k \gamma_{k+1}(A_{k+1}\cos(\gamma_{k+1} z_k) - B_{k+1}\sin(\gamma_{k+1} z_k)),\ k=1,2,\ldots,N-1 \tag{19}$$

$$A_N(\gamma_N \cos(\gamma_N h) + \alpha \sin(\gamma_N h)) + B_N(\alpha \cos(\gamma_N h) - \gamma_N \sin(\gamma_N h)) = 0. \tag{20}$$

Eqs. (17)-(20) form a homogeneous system of algebraic equations with $2N$ unknowns A_k, B_k, $k=1,2,\ldots,N$. It has a nontrivial solution if and only if the determinant of the coefficient matrix equals zero.

Similarly, as in the previous case we get the whole family of eigenvalues λ_m, $m=1,2,3,\ldots$. Following the appropriate notation the corresponding eigenfunctions are obtained from the equations analogous to eqs. (14)-(15).

3. INHOMOGENEOUS PERTURBED BOUNDARY PROBLEM

In this section we consider the fact that the inhomogeneous layered medium in all layers has height dependent properties. This results in the refractive index and hence the velocity depends on height. We use the perturbation technique [6-9] based on Skrzypczyk's approach [1-2, 10-12] to find the eigenvalues and the eigenfunctions.

Due to the inhomogeneity of layers, the eq. (2) is now assumed to be the II-order perturbed equation of the form

$$\frac{d^2\varphi^{(i)}}{dz^2} + k_i^2\left(1 + \varepsilon s_1^{(i)}(z) + \varepsilon^2 s_2^{(i)}(z)\right) = \lambda \varphi^{(i)},\ i = 1,2,\ldots,N \tag{21}$$

where $s^{(i)}(z) := \varepsilon s_1^{(i)}(z) + \varepsilon^2 s_2^{(i)}(z)$ determines the height dependent inhomogeneity of the *i-th* layer. If $s^{(i)}(z)=0$ the problem reduces to that of the homogeneous layered medium discussed above. Following [13-14] we assume the eigenvalue problem will be considered in the generalized sense in $R_{\varepsilon 2}$. Assume further that $\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2$, $\varphi^{(i)} = \varphi_0^{(i)} + \varepsilon\varphi_1^{(i)} + \varepsilon^2\varphi_2^{(i)}$, where $\lambda_0, \lambda_1, \lambda_2, \varphi_0^{(i)}, \varphi_1^{(i)}, \varphi_2^{(i)} \in R$. Comparing coefficients of like powers on both sides of eq. (21) one can obtain the unperturbed equations

$$\frac{d^2\varphi_0^{(i)}}{dz^2} + k_i^2\varphi_0^{(i)} = \lambda_0\varphi_0^{(i)} \tag{22}$$

$$\frac{d^2\varphi_1^{(i)}}{dz^2} + k_i^2\left(\varphi_1^{(i)} + \varphi_0^{(i)} s_1^{(i)}(z)\right) = \left(\lambda_0\varphi_1^{(i)} + \lambda_1\varphi_0^{(i)}\right) \tag{23}$$

$$\frac{d^2\varphi_2^{(i)}}{dz^2} + k_i^2\left(\varphi_2^{(i)} + \varphi_1^{(i)} s_1^{(i)}(z) + \varphi_0^{(i)} s_2^{(i)}(z)\right) = = \lambda_0\varphi_2^{(i)} + \lambda_1\varphi_1^{(i)} + \lambda_2\varphi_0^{(i)} \tag{24}$$

Similarly to the unperturbed case we start solving system (22)-(24) with eq. (22) written in the transformed form

$$\frac{d^2\varphi_0^{(i)}}{dz^2} + \gamma_i^2\varphi_0^{(i)} = 0 \tag{25}$$

where $\gamma_i^2 = k_i^2 - \lambda_0$, $i=1,2,\ldots,N$.

Since the eigenvalue problem (25) is of the Sturm-Liouvile operator type, it has infinite set of eigenvalues, say $\{\gamma_{i1}, \gamma_{i2}, \gamma_{i3}, \ldots, \gamma_{in}, \ldots\}$, $i=1,2,\ldots,N$. So further we denote $\gamma_{ip}^2 = k_i^2 - \lambda_{0p}$, $i=1,2,\ldots,N$, $p=1,2,3,\ldots$. Let $\{\varphi_{01}^{(i)}, \varphi_{02}^{(i)}, \varphi_{03}^{(i)}, \ldots, \varphi_{0n}^{(i)}, \ldots\}$ be the corresponding eigenfunctions. After some calculations

$$\lambda_{1p} = \sum_{i=1}^{N} k_i^2 \left\langle s_1^{(i)}(z)\varphi_{0p}^{(i)}, \varphi_{0p}^{(i)} \right\rangle_i \quad \text{for } p=1,2,3,\ldots \tag{26}$$

$$\lambda_{2p} = \sum_{i=1}^{N} \left\langle k^{(i)^2} s_1^{(i)}(z)\varphi_{1p}^{(i)}, \varphi_{0p}^{(i)} \right\rangle_i + \sum_{i=1}^{N} \left\langle k^{(i)^2} s_2^{(i)}(z)\varphi_{0p}^{(i)}, \varphi_{0p}^{(i)} \right\rangle_i \quad \text{for } p=1,2,3,\ldots \tag{27}$$

The corresponding eigenvectors are as follows

$$\Phi_{1p} = \sum_{i \neq p} \frac{\chi_{pi}}{\lambda_{0i} - \lambda_{0p}} \Phi_{0i}\ ,\quad \Phi_{2p} = \sum_{i \neq p} \frac{\eta_{pi}}{\lambda_{0i} - \lambda_{0p}} \Phi_{0i} \tag{28}$$

where

$$\Phi_{mp}(z) = \begin{cases} \varphi_{mp}^{(1)}(z) & 0 \le z < z_1 = d_1 \\ \varphi_{mp}^{(i)}(z) & z_{i-1} \le z < z_i,\ i = 2,3,\ldots,N-1; \\ \varphi_{mp}^{(N)}(z) & z_{N-1} \le z \le z_N = h \end{cases}, \tag{29}$$

$m=0,1,2;\ p=1,2,3,\ldots$.

The coefficients χ_{pi} and η_{pi} are calculated from Fourier decomposition of known functions.

4. NUMERICAL EXAMPLE

Consider two layers medium with rigid upper side of cylindrical domain. Let's assume further that $h=1300$ [m], $d_i=750$ [m], $c_1=1500$ [m/s], $c_2=1495$ [m/s], $\omega=376.991$ [rad/s], $\rho_1=1.0*103$ [kg/m^3], $\rho_2=1.0*103$ [kg/m^3], *cf.* [5, 13-14].

Let's assume that in the considered example one has a linear perturbation on the index of refraction obtained from

$$s_1(z) = \begin{cases} 0 & 0 \le z \le d_1 \\ d_1 - z & d_1 < z \le h \end{cases},\quad s_2(z) = 0\ .$$

If we solve numerically eqs. (10)-(13) we get the whole family of eigenvalues m, m=1,2,3,.. see Fig. 3. The corresponding eigenfunctions are obtained from eqs. (14)-(15), (compare Fig. 2).

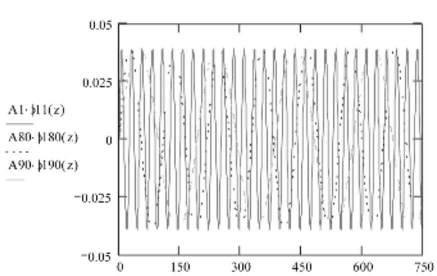

Figure 2.
1-st, 80-th, 90-th eigenfunctions $\varphi^{(1)}$(x)

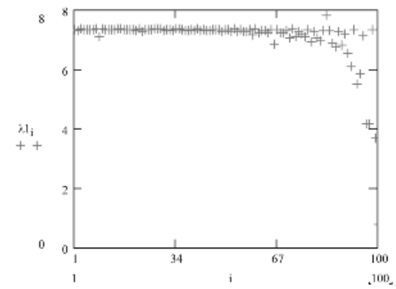

Figure 3.
First perturbations λ_1 of eigenvalues

6. CONCLUSIONS

We have considered the effects of height dependent density on the eigenfunctions and eigenvalues of the equation resulting from the Helmholtz equation satisfied by the acoustic pressure in a layered medium of finite height. In addition to the more convenient boundary condition assuming the boundary to be rigid, a more general boundary condition, namely the reflecting type boundary condition has been considered.

We have used the special methodology of perturbation method to obtain the eigenvalues and eigenfunctions of the inhomogeneous layered model in which the lower layer is assumed to have position dependent density and therefore the wave number. We have computed the eigenvalues and plotted eigenfunctions in cases of interest using the example boundary conditions. The example presented in part 5 is that of a linear dependence of the wave number on the height. In this case the first correction in the eigenvalues is not significant and the eigenmodes remain close to the ones in the unperturbed case. However, in some cases the effects of the perturbation are more significant. Fig. 2 shows the three eigenfunctions in this case. In Fig. 3 one can see the first perturbations in the case of a linear perturbation in the wave number.

Calculations with the use of new perturbation numbers lead to applications which are mathematically equivalent with II order approximations in classical perturbation methods. Advantages of the new algebraic system are as follows:

- we can omit all complex analytical calculations which are typical for expanding approximated values of solutions in infinite series. It works for expanding unknown solutions as well as for perturbed coefficients of the problem;
- we get a great simplification of all arithmetic calculations which appear in analytical formulation and analysis of the problem;
- most of known numerical algorithms can be simply adapted for the new algebraic system without any serious difficulties.

The model can be simply generalized to more layers than two. The new methodology can be applied to any complicated boundary problems with different types of boundary conditions.

BIBLIOGRAPHY

[1] *Skrzypczyk J.*; II-Order Perturbation Methods in Mechanics – New Algebraic Methodology. Proceedings of I Congress of Polish Mechanics, Warszawa 28-31 sierpnia 2007; Warszawa, J. Kubik, W. Kurnik, W. K. Nowacki (Eds.), 2007; CD 166.

[2] *Skrzypczyk J.*; II-Order Perturbation Methods in Mechanics. Modelowanie Inżynierskie 3, Gliwice, 2007; p.111-118.

[3] *Buchanan J.* et. al.; Marine Acoustics – Direct and Inverse Problems. SIAM; 2004.

[4] *Eds. Woźniak Cz., Świtka R., Kuczma M.*; Selected Topics In the Mechanics of Inhomogeneous Media. University of Zielona Góra, Zielona Góra; 2006.

[5] *Zaman F. D.; Al-Muhiameed Z. I. A.*; Acoustic waves in a Layered Inhomogeneous Ocean. Applied Acoustics, 61, 2000; 427-440.

[6] *Kato T.*; Perturbation Theory for Linear Operators. Springer-Verlag, Berlin Heidelberg New York; 1966.

[7] *Nayfeh A. H.*; Perturbation Methods. J. Wiley & Sons; 1976.

[8] *Awrejcewicz J., Krysko V. A.*; Introduction to Modern Asymptotic Methods. WN-T, Warszawa; 2004.

[9] *Shivamoggi B. K.*; Perturbation Methods for Differential Equations. Birkhäuser, Boston – Basel – Berlin; 2002.

[10] *Skrzypczyk J.*; Perturbation methods – New Arithmetic. Zesz. Nauk. Pol. Śl., ser. Bud., Gliwice, 2003; p. 391-398.

[11] *Skrzypczyk J.*; Perturbation Methods – New Arithmetics. Zeszyty Naukowe Katedry Mechaniki Stosowanej Politechniki Śląskiej, vol. 23, Gliwice, 2004; p. 363-368.

[12] *Skrzypczyk J.*; Perturbation Methods – New Algebraic Methodology. Proc. of CMM-2005 – Computer Methods in Mechanics, June 21-24, 2005, Częstochowa, Poland.

[13] *Skrzypczyk J.. Winkler-Skalna A.*; Sound Wave Propagation Problems New Perturbation Methodology. Arch. of Acoustic, 31 (N.4) Supl., Warszawa, 2006; p. 115-122.

[14] *Skrzypczyk J., Winkler-Skalna A.*; Sound Wave Propagation Problems – New Perturbation Methodology. Proc. of Int. Conf. New Trends In Statics And Dynamics of Buildings, October 2006, Faculty of Civil Eng. SUT Bratislava, Slovakia; p. 97-100.