II-ORDER PERTURBATION METHODS NEW COMPUTATIONAL METODOLOGY

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Abstract
The aim of the paper is to present a new algebraic system with specifically defined addition and multiplication operations. The new numbers, called II-order perturbation numbers are introduced. Classical II-order perturbation problems can be solved in the new algebraic system as easily as usual problems of applied mathematics, theoretical physics and techniques. Static perturbation problems of a simple frame are discussed as well as dynamical vibration problems.

Streszczenie
W pracy przedstawiony jest nowy system algebraiczny, ze szczególnie zdefiniowanymi operacjami dodawania i mnożenia. Wprowadzone nowe liczby nazwano liczbami perturbacyjnymi II rzędu. W nowym systemie algebraicznym i zaproponowanej metodologii klasyczne zagadnienia perturbacyjne II-rzędu mogą być rozwiązywane tak łatwo, jak zwykłe zagadnienia matematyki stosowanej, fizyki teoretycznej i techniki. W przykładzie pokazano zastosowania do zagadnień perturbacyjnych w zadaniu dla prostej ramy, zarówno z punktu widzenia problemów statyki, jak i dynamiki.

Keywords: **Perturbation numbers; Perturbation methods; Vibrations; Frames; Perturbed parameters.**

1. INTRODUCTION

Theory of perturbations first appeared in one of the oldest branches of applied mathematics – celestial mechanics. The scope of perturbation theory at the present time is much broader than its applications to celestial mechanics, but the main idea is the same. Theory of perturbations is a part of science of the great theoretical and practical meaning. It begins in 1926/27 with papers of Rayleigh and Schrödinger. Now perturbation theory has got a bibliography which included thousands of positions and is still in developing. [1, 2, 3]

One can begin with a simply solvable problem, called the unperturbed problem and using the solution of this problem as an approximation we go to the solution of a more complicated problem that differs from the basic one only by some small terms in the equations. Then one looks for a series of successive approximations to this initial solution, most often in the form of a power series in a small quantity called the perturbation parameter.

Generally the basic problem of perturbation theory is to answer how much the solution, say $\boldsymbol{x_0}$ changes if coefficient matrix $\boldsymbol{A}$ takes a new value $\boldsymbol{A}+\varepsilon\boldsymbol{B}$, where ε is called a small parameter and $\boldsymbol{B}$ is the perturbation. It is often convenient to seek the solution in the form of a series of homogeneous terms, that is, solutions of the form

$$\boldsymbol{x}:=\boldsymbol{x}_0+\varepsilon\boldsymbol{x}_1+\varepsilon^2\boldsymbol{x}_2+\varepsilon^3\boldsymbol{x}_3+\ldots\ldots\ldots \quad (1)$$

If we restrict our considerations to first three terms in (1) we have perturbation method of the 2-nd order. In perturbation method applications a serious difficulty is a necessity of a large amount of analytical calculations to obtain series expansion. As a result we obtain a set of classical problems which are usually easier to solve numerically, cf. [2, 3].

In the paper special numbers, called further perturbations numbers of the II-order (II-order PN's), similar to perturbation numbers defined as in the author's earlier papers are used, cf. [6-12].

Perturbation value functions are defined for II-order perturbation arguments as extension of classical elementary and trigonometric functions. [6-8, 11-12]

Calculations with the use of new II-order perturbation numbers lead to applications which are mathematically equivalent to II order approximations in classical perturbation methods.

New mathematical formalism is applied to classical perturbation problems arising in theoretical mechanics. Static perturbation problems of a simple frame with perturbed coefficients and loads (systems of perturbed linear algebraic equations) are discussed as well as dynamical vibration problems (perturbed generalized eigenvalue problem). The advantages of the new methodology are presented in analytical calculations and in special numerical procedures dedicated to linear systems of perturbed equations and eigenvalue problems. The new technique can be used as the basis for equations with mixed-type variables analysis in the situation when all parameters of equations are perturbed.

2. ALGEBRAIC SYSTEM OF PERTURBATION NUMBERS

Let's define a new number called further a 2-nd order perturbation number as an ordered 3-tuple of real numbers $(x,y,z) \in R^3$. The set of perturbation numbers is denoted by $R_{\varepsilon 2}$. The first element x of the 3-touple (x,y,z) is called the main value $mv(.)$, the second y – the perturbation of the 1-st order – $pv1(.)$ and the third z – the perturbation of the 2-nd order value – $pv2(.)$. [11-12]

Let $a, a_1, a_2, a_3 \in R_{\varepsilon 2}$ denote any perturbation numbers of the 2-nd order and $a := (x,y,z)$, $a_i := (x_i, y_i, z_i)$, $x, y, z, x_i, y_i, z_i \in R$, $i = 1,2,3$. It is called that two perturbation numbers $a_1 \equiv a_2$ are equal if and only if (iff): $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

In the set $R_{\varepsilon 2}$ the addition ($+_{\varepsilon 2}$) and multiplication ($\bullet_{\varepsilon 2}$) are defined as follows:

$$a_1 +_{\varepsilon 2} a_2 := (x_1 + x_2, y_1 + y_2, z_1 + z_2) \tag{2}$$

$$a_1 \bullet_{\varepsilon 2} a_2 := (x_1 x_2, x_1 y_2 + x_2 y_1, x_1 z_2 + z_1 x_2 + y_1 y_2) \tag{3}$$

The specially defined neutral addition element $0_{\varepsilon 2} := (0,0,0)$ and neutral multiplication element $1_{\varepsilon 2} := (1,0,0)$ will be further in use.

The field $R_{\varepsilon 2}$ as defined above doesn't contain the field of real numbers R. We can show that real numbers can be considered as some elements of the field $R_{\varepsilon 2}$ with all classical addition and multiplication formulas and neutral elements of addition and multiplication, cf. [6-8], [11-12].

The map $j: R \to R_{\varepsilon 2}$, $j(x) := (x,0,0)$ for each $x \in R$, is the injection of the algebraic system of real numbers R into the algebraic system $R_{\varepsilon 2}$. It's the single-valued mapping and preserves corresponding algebraic operations and neutral elements of addition and multiplications.

3. SIMPLIFIED NOTATION FOR PERTURBATION CALULATIONS

Notice, that since $j(.)$ is the injection then each perturbation number of the form $(a,0,0)$, $a \in R$ can be identified with a real number a. We use this insight to simplify a notation for perturbation operations. Denote by $\varepsilon := (0,1,0)$ and by $\eta := (0,0,1)$.

As usually, we abbreviate further $a_1 \bullet_{\varepsilon 2} a_2$ as $a_1 a_2$. Let's assume that the perturbation number $(x,0,0)$ is identified with $x \in R$, $(y,0,0)$ with $y \in R$ and $(z,0,0)$ with the real z. Then for any $(x,y,z) \in R_{\varepsilon 2}$, we have

$$\begin{aligned}(x,y,z) = {} & (x,0,0) + (y,0,0)\,(0,1,0) + (z,0,0)\,(0,0,1) = \\ & = j(x) + \varepsilon j(y) + \eta j(z) = x + \varepsilon y + \eta z\end{aligned} \tag{4}$$

If $y=0$ and $z=0$, then (x,y,z) is the real number x.

From multiplication (3) formula it follows, that

$$\varepsilon^2 := \varepsilon\,\varepsilon = (0,1,0)\,(0,1,0) = (0,0,1) = \eta$$

and according to simplified notation we have $\varepsilon^2 = \eta$. Similarly

$$\varepsilon\eta = (0,0,0),\ \eta^2 = (0,0,0),\ \varepsilon^3 = (0,0,0),$$

and simply $\varepsilon^3 = 0$.

They are called further 2-nd order perturbation numbers and are the ordered 3-touples of real numbers $(x,y,z) \in R^3$, which can be written in the simplified form: $a := x + \varepsilon y + \eta z = x + \varepsilon y + \varepsilon^2 z$.

Supporters of „usual" perturbation methods can use new numbers as simply as real numbers with usual operations: addition, subtraction, multiplication and division. The symbol ε can be treated as a 2-nd order small parameter, with the property $\varepsilon^3=0$.

Relations for usual arithmetic operations take the following form:

$$a_1+a_2 := x_1+x_2+\varepsilon(y_1+y_2)+\varepsilon^2(z_1+z_2) \quad (5)$$

Notice $(-a):=(-x,-y,-z)$. The subtraction can be defined as

$$a_1-a_2 = a_1+(-a_2), \text{ then}$$

$$a_1-a_2 := x_1-x_2+\varepsilon(y_1-y_2) +\varepsilon^2(z_1-z_2) \quad (6)$$

$$\alpha a := \alpha x+\varepsilon(\alpha y)+\varepsilon^2(\alpha z), \quad for\ \alpha\in R^1 \quad (7)$$

$$a_1a_2 := x_1x_2+\varepsilon(x_1y_2+x_2y_1)+\varepsilon^2(x_1z_2+z_1x_2+y_1y_2) \quad (8)$$

The inverse of the perturbation number $a=x+\varepsilon y+\varepsilon^2 z$ is defined as the perturbation number $a^{-1}=x_1+\varepsilon y_1+\varepsilon^2 z_1$ such that

$$aa^{-1}= (x+\varepsilon y+\varepsilon^2 z)(x_1+\varepsilon y_1+\varepsilon^2 z_1)=(1,0,0),\ x,y,z,x_1,y_1,z_1\in \mathrm{R}.$$

Notice further that

$$x_1+\varepsilon y_1+\varepsilon^2 z_1=(x,y,z)^{-1}=\left(\frac{1}{x},-\frac{y}{x^2},-\frac{z}{x^2}+\frac{y^2}{x^3}\right), \quad x\neq 0 \quad (9)$$

The formula for division can be simply introduced as

$$a_1/a_2=\left(x_1+\varepsilon y_1+\varepsilon^2 z_1\right)\Big/\left(x_2+\varepsilon y_2+\varepsilon^2 z_2\right)=$$

$$=\frac{x_1}{x_2}+\varepsilon\left(\frac{y_1}{x_2}-\frac{x_1y_2}{x_2^2}\right)+\varepsilon^2\left(-\frac{x_1z_2}{x_2^2}+\frac{x_1y_2^2}{x_2^3}-\frac{y_1y_2}{x_2^2}+\frac{z_1}{x_2}\right), \quad x_2\neq 0$$

4. EXTENDED ε-FUNCTIONS

Perturbation value functions are defined for perturbation arguments as extensions of classical elementary and trigonometric functions. Properties of ε-functions are analyzed in details in [6-8, 11-12].

Let $D\subset R_{\varepsilon 2}$ be an arbitrary subset. Suppose that we have a rule $f_{\varepsilon 2}$ which assigns to each element $z\in D$ exactly one element w of $R_{\varepsilon 2}$. Then we say that $f_{\varepsilon 2}(.)$ is an extended function defined on D with values in $R_{\varepsilon 2}$. We will denote that function as $f_{\varepsilon 2}:D\rightarrow R_{\varepsilon 2}$ or $w = f_{\varepsilon 2}(z)$ or simplified $w = \varepsilon 2\text{-}f(z)$.

To illustrate how we can construct generalizations of usual real functions we use a simple function. We are discussing now an extension of a simple exponential function $exp(x)$, $x\in R$. With polynomials and rational functions it is one of the simplest elementary functions. How can we understand the notion $exp(z)$, where $z=x+\varepsilon y\in R_{\varepsilon 2}$?

Notice that we can expand $exp(x)$, $x\in R$ into a classical series

$$exp(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......=\sum_{k=0}^{\infty}\frac{x^k}{k!}, \qquad x\in R \quad (10)$$

which is convergent for all $x\in R$. Define the new function $exp_{\varepsilon 2}(a)$, for $a=x+\varepsilon y+\varepsilon^2 z\in R_{\varepsilon 2}$ as

$$exp_{\varepsilon 2}(a):=1+\frac{a}{1!}+\frac{a^2}{2!}+\frac{a^3}{3!}+.......=\sum_{k=0}^{\infty}\frac{a^k}{k!}, \qquad a\in R_{\varepsilon 2} \quad (11)$$

Following equations (10) and (11) we get

$$exp_{\varepsilon 2}(a)=\left(1+\varepsilon y+\varepsilon^2\left(z+\frac{y^2}{2}\right)\right)exp(x) \qquad a\in R_\varepsilon \quad (12)$$

We can prove the generalized convergence of the sequence (12) for every $a\in R_{\varepsilon 2}$. Additionally we have

$$j(exp(x)) =(exp(x),0,0)=exp_{\varepsilon 2}(x),$$

which proves that the new function $exp_{\varepsilon 2}(.)$ is the extension of the real function exp(x).

5. EXAMPLE

New mathematical formalism is applied to classical perturbation problems arising in theoretical mechanics. Static perturbation problems of a simple frame, see Fig.1, are discussed as well as dynamical stability problems, technical details cf. [1].

Balance equations take the classical form $Kq=F$, where

$$\frac{EJ}{l^3}\begin{bmatrix}12+\lambda^2 & 0 & -6l & -\lambda^2\\ 0 & 12+\lambda^2 & 6l & 0\\ -6l & 6l & 8l^2 & 0\\ -\lambda^2 & 0 & 0 & \lambda^2\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}=\begin{bmatrix}0\\ 6pl\\ \frac{pl^2}{12}\\ 0\end{bmatrix}, \ \lambda^2=\frac{Al^2}{J} \quad (13)$$

Assume $\lambda=80$, $l=5$, $p=4.157$, all values are dimensionless. For these nominal quantities we assume that: 1-st order perturbations of all nonzero elements are random and about up to ±10% of the nominal value, 2 nd order perturbations of all nonzero elements are random and about ±1% of the nominal value. The signs of perturbations are random. Nonzero numerical values for calculations are taken as:

$$k_{11}=6412.0-\varepsilon 641.17-\varepsilon^2 53.22,\ k_{13}=-30.0-\varepsilon 2.81+\varepsilon^2 0.19$$

$$k_{14}=-6400.0+\varepsilon 303.48-\varepsilon^2 31.17,$$

$k_{22} = 6412.0 + \varepsilon 129.98 + \varepsilon^2 50.22,\ k_{23} = 30.0 + \varepsilon 0.09 - \varepsilon^2 0.06$
$k_{33} = 200.0 - \varepsilon 16.42,\ k_{44} = -6400.0 + \varepsilon 303.48$

$$\boldsymbol{F} = \left[0.000 \quad 124.703 + \varepsilon 2.293 + \varepsilon^2 0.029 \quad 8.660 + \varepsilon 0.652 + \varepsilon^2 0.086 \quad 0.000\right]^T$$

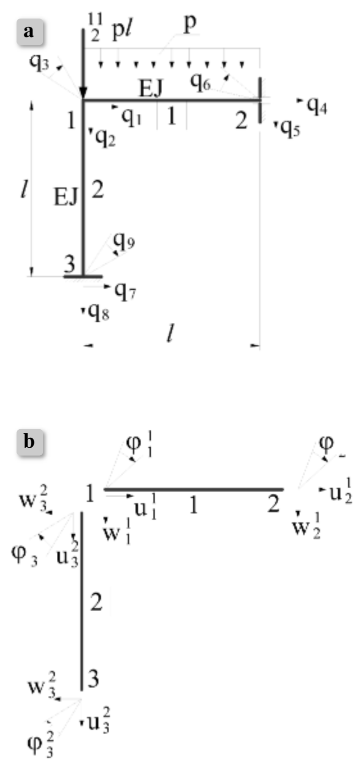

Figure 1.
Scheme of the frame [1]

Numerical calculations in the new arithmetic are very easy to programming, almost with same complexity as for real or complex numbers. Calculations in the example were made with the single precision accuracy with the use of modification of the standard Gauss elimination method with pivoting applied to the main part of the matrix $\boldsymbol{K}$. The numerical results are as follows:

$$\boldsymbol{q} = \frac{l^3}{EJ}\begin{bmatrix} 0.162 - \varepsilon 1.465 - \varepsilon^2 15.853 \\ 0.019 + \varepsilon 0.009 - \varepsilon^2 0.011 \\ 0.0647 - \varepsilon 0.209 + \varepsilon^2 0.000 \\ 0.162 - \varepsilon 1.475 + \varepsilon^2 15.946 \end{bmatrix}$$

The matrix $\boldsymbol{K}\text{-}\sigma^2\boldsymbol{K}_G^0$ must be weakly positively defined [6-9] to assure a sufficient condition for stability of the considered frame, where

$$\boldsymbol{K}_G{}^0 = \begin{bmatrix} \frac{6}{5} & 0 & -\frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{10} & 0 & \frac{2l^2}{15} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad \sigma^2 = \frac{Sl^2}{EJ}$$

For $\sigma^2=0$ the above matrix is the stiffness matrix $\boldsymbol{K}$, which has the property of positivity. But the matrix can lose that property if the following equality is satisfied $det_{\varepsilon 2}(\boldsymbol{K}\text{-}\sigma^2\boldsymbol{K}_G^0) = 0$, i.e.

$$\frac{3}{20}\left(12 + \lambda^2\right)\sigma^4 - \frac{2}{5}\left(192 + 25\lambda^2\right)\sigma^2 + 12\left(24 + 5\lambda^2\right) = 0 \quad (14)$$

The value of $\lambda=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2 \in R_\varepsilon$. Eq. (14) has two solutions

$$\sigma_{1,2}{}^2 = \frac{4}{3}\frac{192 + 25\lambda^2 \mp 2\sqrt{5976 + 1455\lambda^2 + 100\lambda^4}}{12 + \lambda^2} \quad (15)$$

and for arbitrary λ, say $\lambda=80+\varepsilon 0.8+\varepsilon^2 20.08$, we obtain ε-values

$\sigma_1{}^2=6.663858+\varepsilon 5.612860E\text{-}05+\varepsilon^2 4.770570E\text{-}06$

$\sigma_2{}^2=59.957890+\varepsilon 8.406466E\text{-}04+\varepsilon^2 7.148685E\text{-}05.$

For further details about more complex perturbation calculation and functions see [11-12].

6. CONCLUSIONS

Calculations with the use of new perturbation numbers lead to applications which are mathematically equivalent to II order approximations in classical perturbation methods. Benefits of the new algebraic system are as follows: we can omit all complex analytical calculations which are typical for expanding approximated values of solutions in infinite series. It works for expanding unknown values – solutions as well as for perturbed coefficients of the problem; we get a great simplification of all arithmetic calculations which appear in analytical formulation and analysis of the problem; most of known numerical algorithms can be simply adapted for the new algebraic system without any serious difficulties.

With the new algebraic system we get a set of very simple and useful mathematical tools which can be easily used in analytical and computational parts of analysis of complex perturbation problems.

Examples of applications for classical problems of computational mechanics are presented including a detailed numerical analysis: the elastic frame with perturbed coefficients and loads and perturbed stability.

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